

Stochastic Modeling of the Thermally Induced Atmospheric Flow at Mesoscale

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Abstract. This paper presents a three-dimensional stochastic linear model of the atmospheric flow induced by the variability of heat flux over land surface. The primitive equations relating perturbation terms of wind field, geopotential and buoyancy are formulated as a system of stochastic partial differential equations and solved analytically. The solution is based on spectral representations of the homogeneous random fields. The flow intensity is found to be proportional to the standard deviation of the heat flux into the atmosphere. The intensity of the vertical motion becomes more sensitive to the differential heating with a larger length scale as altitude goes higher. Stability and synoptic wind inhibit the development of the flow. The proposed theory improves the understanding of the role that heterogeneous land surface plays in atmospheric circulations at the mesoscale.

Sommario. Questo lavoro presenta un modello stocastico lineare del moto atmosferico tridimensionale indotto dalla variabilità del flusso di calore sulla superficie del terreno. Le equazioni primitive che legano i termini perturbativi del campo di vento, del geopotenziale e del parametro di galleggiamento sono formulate come un sistema di equazioni stocastiche alle derivate parziali che vengono risolte analiticamente. Tale soluzione è basata su rappresentazioni spettrali di campi aleatori omogenei. L'intensità del moto risulta essere proporzionale alla deviazione standard del flusso di calore diretto verso l'atmosfera. L'intensità del moto in direzione verticale appare più sensibile al riscaldamento differenziale, con scale spaziali più grandi al crescere dell'altitudine. La stabilità ed il flusso sinottico tendono ad inibire lo sviluppo del moto. La teoria qui proposta migliora la comprensione del ruolo che la superficie eterogenea del terreno gioca nella circolazione atmosferica alla meso-scala.

Key words: Surface heating, Mesoscale flow, Stochastic approach, Hydrometeorology.

1. Introduction

Predictions with large scale atmospheric models depend heavily on how accurately subgrid scale variabilities are characterized. The theoretical approaches of Rotunno [1], Dalu and Pielke [2] and Dalu *et al.* [3], and the numerical simulations of Avissar and Pielke [4] have suggested that the intensity of the mesoscale circulations induced by the thermal heterogeneities of the land surface could be as strong as the sea breeze, implying a significant contribution to the total energy and mass transport. Rotunno [1] presented a two-dimensional linear land and sea breeze model to investigate the atmospheric response to the periodic diurnal cycle of heating and cooling over contrasting land and sea. Dalu and Pielke [2] and Dalu *et al.* [3] extended Rotunno's work to study the effects of the non-periodic forcing and of friction.

Three major improvements over previous analytical work have been achieved in the stochastic linear theory presented in this paper. First, a three-dimensional model is adopted. All the aforementioned theoretical studies utilize two-dimensional models. Second, the thermal inhomogeneity of land surface in our model is characterized by a two-dimensional random field that realistically represents the complexity of natural landscape. This is in contrast to the over-simplified landscapes, in the form of periodic warm-cool stripes, of the existing two-

dimensional analytical models. Third, the effect of synoptic wind is studied quantitatively. To our knowledge, this issue has not been addressed in previous analytical work.

2. Stochastic Modeling of 3D Atmospheric Flow

2.1. GOVERNING EQUATIONS

The two-dimensional linear land and sea breeze model of Rotunno [1] and Dalu and Pielke [2] is generalized to describe the flow driven by the gradient of diabatic heating due to the variability of the landscape in a three-dimensional domain in the presence of a constant synoptic wind,

$$\frac{\partial u}{\partial t} + u_0 \frac{\partial u}{\partial x} - fv = -\frac{\partial \phi}{\partial x} - \alpha u \quad (1)$$

$$\frac{\partial v}{\partial t} + u_0 \frac{\partial v}{\partial x} + fu = -\frac{\partial \phi}{\partial y} - \alpha v \quad (2)$$

$$\frac{\partial w}{\partial t} + u_0 \frac{\partial w}{\partial x} - b = -\frac{\partial \phi}{\partial z} - \alpha w \quad (3)$$

$$\frac{\partial b}{\partial t} + u_0 \frac{\partial b}{\partial x} + N^2 w = Q(x, y, z, t) - \alpha b \quad (4)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (5)$$

where b is the buoyancy, ϕ is the geopotential, f is the Coriolis parameter, $\alpha^{-1} \sim O(T_0)$ is the damping time scale due to friction [2], N is the Brunt-Väisälä frequency, and Q is the buoyancy source due to the surface sensible heat flux. The vertical coordinate z is taken to be positive upward.

2.2. STATISTICAL DESCRIPTION OF THE DIABATIC HEATING

Consider a mesoscale domain with the sensible heat flux at the surface is represented as a two-dimensional homogeneous random field. The diabatic heating is further assumed to decay upward in an exponential fashion with a constant e-folding height h . The exponential function has been used in the theoretical study [1] and is qualitatively consistent with numerical simulations of the turbulent heat flux, e.g. [6]. The diabatic heating has a diurnal cycle which follows closely the insolation curve $I(t)$. This is due to the fact that the turbulent heat flux H_t is almost in-phase with solar radiation as corroborated by many observational studies, e.g. [5]. Hence the buoyancy source Q due to the diabatic heating can be written as

$$Q = \hat{Q}(x, y) \exp\left(-\frac{z}{h}\right) I(t) \quad (6)$$

where \hat{Q} is a random field characterizing the thermal variability of the land surface that will be specified in section 3. The insolation curve $I(t)$ is

$$I(t) = \begin{cases} \sin(\Omega t) , & 0 \leq t \leq T_0/2 \quad (\text{day - time}) \\ 0 & , T_0/2 < t \leq T_0 \quad (\text{night}) \end{cases} \quad (7)$$

where Ω is the rotation rate of the Earth, and T_0 is one day.

The buoyancy source Q is related to the turbulent heat flux H_t through

$$Q = \frac{g}{\rho C_P \theta_0 h} H_t \quad (8)$$

where ρ is the air density, θ_0 is the constant reference potential temperature, C_P is the heat capacity of the air at the constant pressure, and g is the gravity.

3. Analytical Solution

3.1. DECOMPOSITION AND THE SPECTRAL REPRESENTATION

Randomness in the buoyancy source Q makes all dependent variables in equations (1) through (5) random. Given the linearity of the governing equations it is safe to assume that if the external forcing of the flow is horizontally homogeneous so will the dependent variables. The first step to the solution of the stochastic partial differential equations is to decompose the dependent variables u, v, w, b, ϕ and the buoyancy source \hat{Q} into their horizontal domain mean $(\bar{\quad})$ and a perturbation term (\prime) around the mean. The linear dynamic system for the state variables u, v, w, b and ϕ also implies that the perturbation terms (\prime) are decoupled from the mean $(\bar{\quad})$. They both satisfy the governing equations (1) through (5). From now on we focus only on the perturbation terms.

Using the spectral representation [7], the perturbation terms u', v', w', b', ϕ' and \hat{Q}' can be expressed as

$$\eta' = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(k_1 x + k_2 y)} dZ_{\eta}(k_1, k_2; z, t) \quad (9)$$

$$\hat{Q}' = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(k_1 x + k_2 y)} dZ_Q(k_1, k_2) \quad (10)$$

where the symbol η stands for any of the state variables u, v, w, b and ϕ . dZ_{η} and dZ_Q are the random variables in the frequency domain (k_1, k_2) corresponding to u', v', w', b', ϕ' and \hat{Q}' in the physical domain (x, y) . dZ_Q has a prescribed spectral density function, $S_Q(k_1, k_2)$, characterizing the thermal variability of the land surface

$$\sigma_Q^2 S_Q(k_1, k_2) = \frac{E[|dZ_Q(k_1, k_2)|^2]}{dk_1 dk_2} \quad (11)$$

where σ_Q is proportional to σ_H according to (8). σ_H is the standard deviation of the turbulent heat flux into the atmosphere resulting from the maximum net solar radiation at the ground. It is a measure of the thermal gradient at the land surface.

3.2. GOVERNING EQUATIONS IN THE FREQUENCY DOMAIN

The governing equations for dZ 's can be readily derived by substituting the equations (9) and (10) into equations (1) through (5).

$$\left(\frac{\partial}{\partial t} + \alpha + ik_1u_0\right)(dZ_u) - f(dZ_v) = -ik_1(dZ_\phi) \quad (12)$$

$$\left(\frac{\partial}{\partial t} + \alpha + ik_1u_0\right)(dZ_v) + f(dZ_u) = -ik_2(dZ_\phi) \quad (13)$$

$$\left(\frac{\partial}{\partial t} + \alpha + ik_1u_0\right)(dZ_w) - (dZ_b) = -\frac{\partial}{\partial z}(dZ_\phi) \quad (14)$$

$$\left(\frac{\partial}{\partial t} + \alpha + ik_1u_0\right)(dZ_b) + N^2(dZ_w) = I(t) \exp\left(-\frac{z}{h}\right)(dZ_Q) \quad (15)$$

$$ik_1(dZ_u) + ik_2(dZ_v) + \frac{\partial}{\partial z}(dZ_w) = 0. \quad (16)$$

3.3. SOLUTION FOR THE dZ 'S

The analytical solution of equations (12) through (16) with the proper initial and boundary conditions is shown below:

(a) when $N \neq f$

$$dZ_u = i \left\{ \frac{fk_2h}{f^2 - k^2h^2N^2} \left[\exp\left(-\frac{z}{h}\right) - \frac{N}{f}kh \exp\left(-\frac{N}{f}kz\right) \right] I_0(t; u_0, \alpha) - \frac{2}{\pi} \frac{kh}{1 - k^2h^2} \int_a^b \cos(zkG(\xi))G(\xi) \frac{\xi k_1hI_s(t, \xi; u_0, \alpha) - fk_2hI_c(t, \xi; u_0, \alpha)}{(\Lambda^2 - \xi^2)\xi} d\xi \right\} dZ_Q \quad (17)$$

$$dZ_v = -i \left\{ \frac{fk_1h}{f^2 - k^2h^2N^2} \left[\exp\left(-\frac{z}{h}\right) - \frac{N}{f}kh \exp\left(-\frac{N}{f}kz\right) \right] I_0(t; u_0, \alpha) + \frac{2}{\pi} \frac{kh}{1 - k^2h^2} \int_a^b \cos(zkG(\xi))G(\xi) \frac{\xi k_2hI_s(t, \xi; u_0, \alpha) + fk_1hI_c(t, \xi; u_0, \alpha)}{(\Lambda^2 - \xi^2)\xi} d\xi \right\} dZ_Q \quad (18)$$

$$dZ_w = - \left\{ \frac{2}{\pi} \frac{k^2h^2}{1 - k^2h^2} \int_a^b \sin(zkG(\xi)) \frac{I_s(t, \xi; u_0, \alpha)}{\Lambda^2 - \xi^2} d\xi \right\} dZ_Q \quad (19)$$

$$dZ_b = \left\{ \left[\exp\left(-\frac{z}{h}\right) + \frac{k^2h^2N^2}{f^2 - k^2h^2N^2} \left(\exp\left(-\frac{z}{h}\right) - \exp\left(-\frac{N}{f}kz\right) \right) \right] I_0(t; u_0, \alpha) - \frac{2}{\pi} \frac{k^2h^2N^2}{1 - k^2h^2} \int_a^b \sin(zkG(\xi)) \frac{I_c(t, \xi; u_0, \alpha)}{(\Lambda^2 - \xi^2)\xi} d\xi \right\} dZ_Q \quad (20)$$

$$dZ_\phi = h \left\{ \frac{-f^2}{f^2 - k^2 h^2 N^2} \left[\exp\left(-\frac{z}{h}\right) - \frac{N}{f} k h \exp\left(-\frac{N}{f} k z\right) \right] I_0(t; u_0, \alpha) + \frac{2}{\pi} \frac{k h}{1 - k^2 h^2} \int_a^b \cos(z k G(\xi)) G(\xi) \frac{\xi^2 - f^2}{\Lambda^2 - \xi^2} \frac{I_c(t, \xi; u_0, \alpha)}{\xi} d\xi \right\} dZ_Q \quad (21)$$

(b) when $N = f$

$$dZ_u = \frac{i}{f(1 - k^2 h^2)} \left\{ \exp\left(-\frac{z}{h}\right) - k h \exp(-z k) \right\} \{k_1 h I_s(t, f; u_0, \alpha) + k_2 h [I_0(t; u_0, \alpha) - I_c(t, f; u_0, \alpha)]\} dZ_Q \quad (22)$$

$$dZ_v = \frac{i}{f(1 - k^2 h^2)} \left\{ \exp\left(-\frac{z}{h}\right) - k h \exp(-z k) \right\} \{k_2 h I_s(t, f; u_0, \alpha) - k_1 h [I_0(t; u_0, \alpha) - I_c(t, f; u_0, \alpha)]\} dZ_Q \quad (23)$$

$$dZ_w = -\frac{k^2 h^2}{f(1 - k^2 h^2)} \left\{ \exp\left(-\frac{z}{h}\right) - \exp(-k z) \right\} I_s(t, f; u_0, \alpha) dZ_Q \quad (24)$$

$$dZ_b = \left\{ \exp\left(-\frac{z}{h}\right) I_0(t; u_0, \alpha) + \frac{k^2 h^2}{1 - k^2 h^2} \left[\exp\left(-\frac{z}{h}\right) - \exp(-z k) \right] [I_0(t; u_0, \alpha) - I_c(t, f; u_0, \alpha)] \right\} dZ_Q \quad (25)$$

$$dZ_\phi = -\frac{h}{1 - k^2 h^2} \left\{ \exp\left(-\frac{z}{h}\right) - k h \exp(-z k) \right\} I_0(t; u_0, \alpha) dZ_Q \quad (26)$$

where

$$[a, b] = [\min\{f, N\}, \max\{f, N\}] \quad (27)$$

$$\Lambda^2 = \frac{f^2 - k^2 h^2 N^2}{1 - k^2 h^2} \quad (28)$$

$$G(\xi) = \sqrt{\left| \frac{N^2 - \xi^2}{f^2 - \xi^2} \right|} \quad (29)$$

$$I_0(t; u_0, \alpha) = \int_0^t I(\tau) \exp[-(\alpha + i k_1 u_0)(t - \tau)] d\tau \quad (30)$$

$$I_s(t, \xi; u_0, \alpha) = \int_0^t I(\tau) \sin[\xi(t - \tau)] \exp[-(\alpha + i k_1 u_0)(t - \tau)] d\tau \quad (31)$$

$$I_c(t, \xi; u_0, \alpha) = \int_0^t I(\tau) \cos[\xi(t - \tau)] \exp[-(\alpha + i k_1 u_0)(t - \tau)] d\tau \quad (32)$$

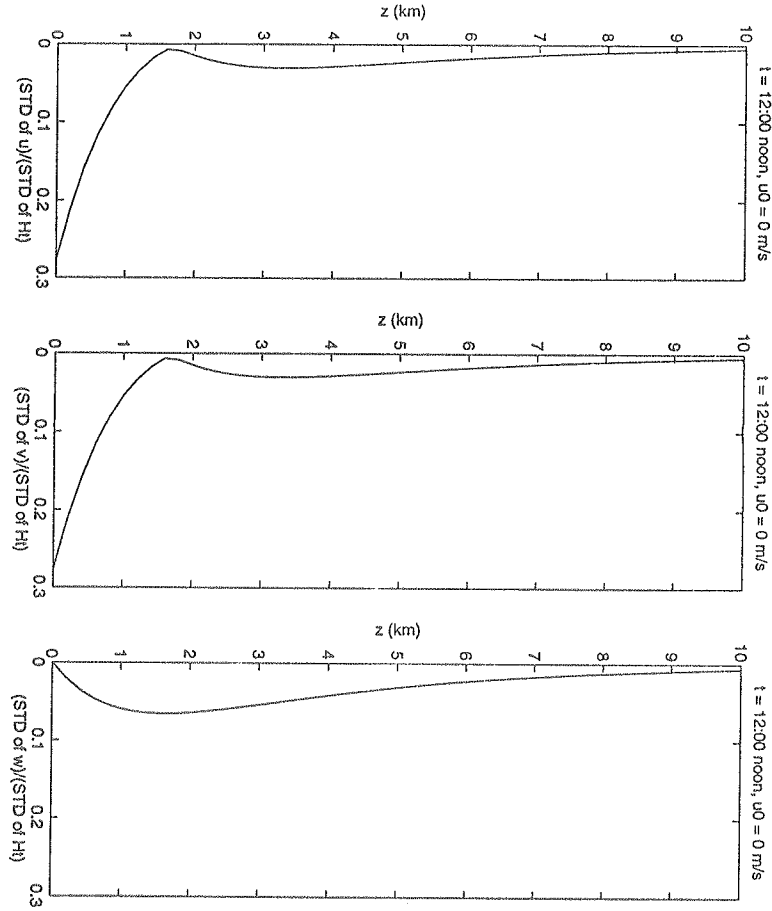


Fig. 1. Vertical profiles of σ_u , σ_v and σ_w (m/s) normalized by σ_H (W/m^2) with $N = 0$ and $u_0 = 0$.

and $I(t)$ given in (7).

For the sake of brevity, we rewrite the solution given in (17) through (26) as

$$dZ_\eta = \Pi_\eta(k_1, k_2; z, t)dZ_Q \quad (33)$$

4. Statistics of the Mesoscale Circulation

In the framework of stochastic analysis the *intensity* of the flow can be quantified by the standard deviation (square root of the variance) σ_u , σ_v and σ_w of the wind velocities u , v and w , respectively. Physically they represent the order of magnitude of the wind velocities. The *horizontal distribution* of the flow can be characterized by length scales. For instance, the length scale of vertical velocity w provides a measure of the size of the circulation cells. In the frequency domain these length scales correspond to wave number(s) around which a large portion of the variance concentrates. The length scale L is equal to the inverse of the wave number k multiplied by 2π .

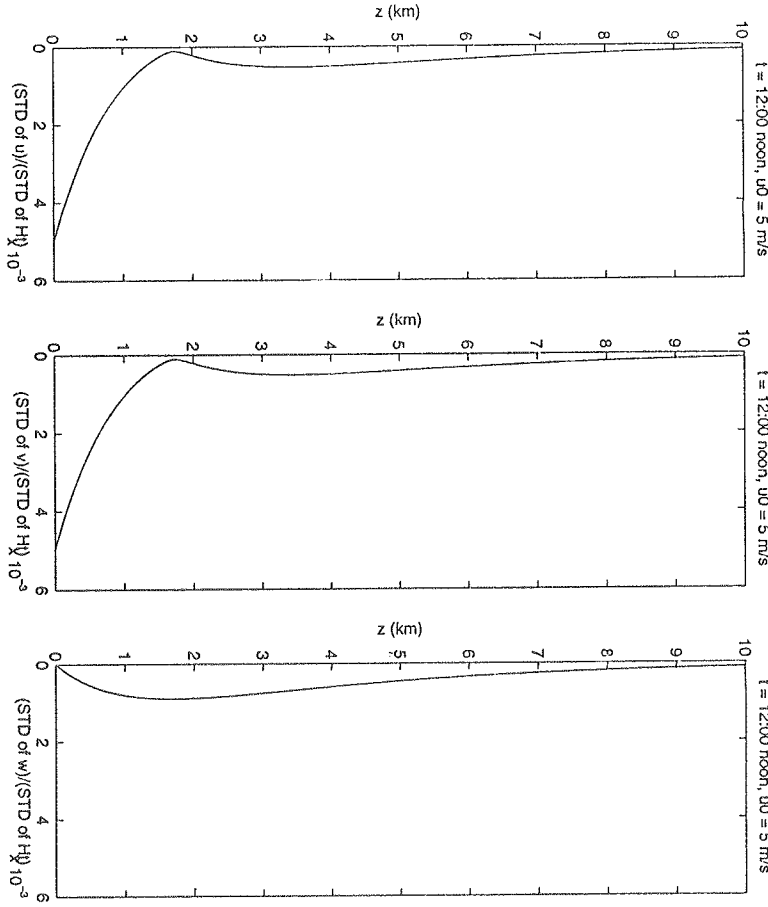


Fig. 2. Same as Figure 1 but $u_0 = 5$ m/s.

4.1. PROPERTIES OF σ_i

The flow intensity is basically determined by the parameters characterizing the synoptic environment and the thermal properties of the landscape. Significant land breeze circulation is expected to be associated with a sufficiently large thermal gradient. Atmospheric stability provides resistance against the upward motion of the air forced by diabatic heating. Strong synoptic wind tends to smooth out the locally generated flow structures. The analytical solution in the previous section enables us to investigate the effects of these parameters quantitatively.

The variances can be computed from equation (33) as

$$\begin{aligned} \sigma_i^2(z, t) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbb{E}[|dZ_i|^2] dk_1 dk_2 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\Pi_i(k_1, k_2; z, t)|^2 \sigma_Q^2 S_Q(k_1, k_2) dk_1 dk_2 \end{aligned} \quad (34)$$

where the index i represents u, v, w . This equation predicts a linear relationship between the flow intensity σ_i and the thermal gradient σ_Q or σ_H . In the following analysis it is convenient to normalize σ_i by σ_H .

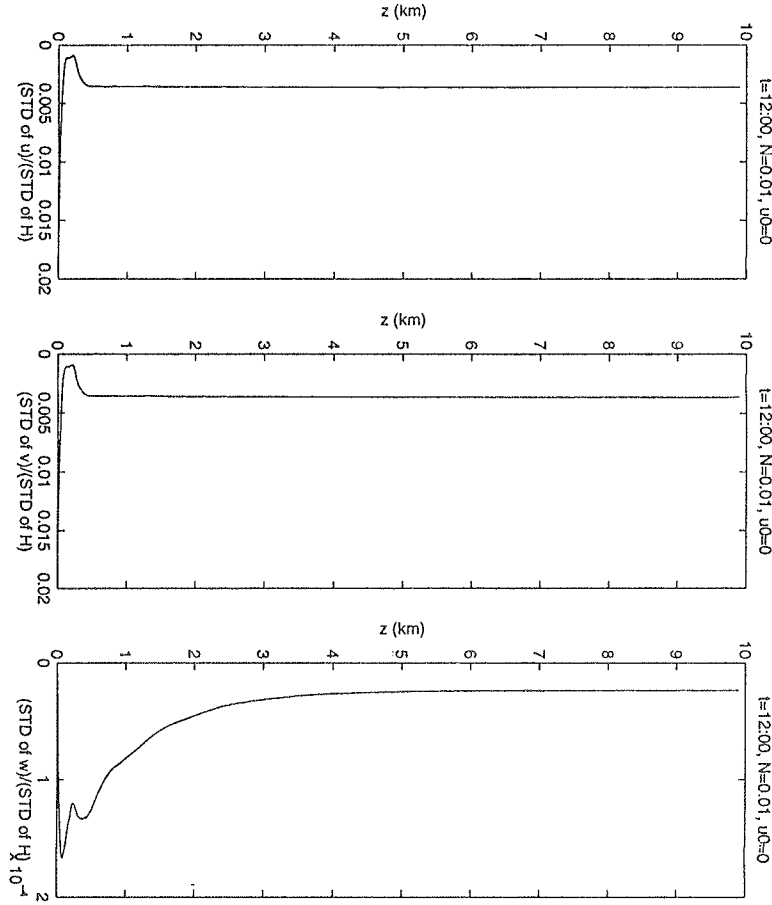


Fig. 3. Same as Figure 1 but $N = 0.01 \text{ s}^{-1}$.

To compute σ_i using (34), the functional form of $S_Q(k_1, k_2)$ needs to be specified. In principle $S_Q(k_1, k_2)$ must be estimated from the measurement of sensible heat flux at the surface over the region of interest. For the purpose of illustration we may use a hypothetical S_Q function of uniform distribution over a finite frequency domain. The thermal variability is assumed to have the length scale ranging from 20 km to 50 km, imposing the highest and lowest wave number cut-off in the (k_1, k_2) domain.

First we investigate the role of synoptic wind u_0 on σ_i under the conditions of neutral stratification ($N = 0$) and no friction ($\alpha = 0$), yielding an upper limit to σ_i . Comparing σ_i with $u_0 = 0$ in Figure 1 with that with $u_0 = 5 \text{ m/s}$ in Figure 2, we see the presence of the synoptic wind strongly inhibits the development of the flow. σ_i decreases by a factor of 30 as the synoptic wind u_0 increases from 0 to 5 m/s. Hence, in an environment with moderate to strong synoptic wind the flow driven by the differential surface heating is negligibly weak.

Stable stratification of the atmosphere affects not only the flow intensity but also the height below which the thermally-induced air flow is active. Compared to Figure 1, where $N = 0$, σ_u and σ_v shown in Figure 3 decrease by a factor of 15 as N increases from 0 to 0.01 s^{-1} . At the same time σ_w decreases by a factor of 300! Also the height of σ_w^{\max} moves from 1800 m down to 100 m. The flow in such an environment is constrained within a shallow layer near

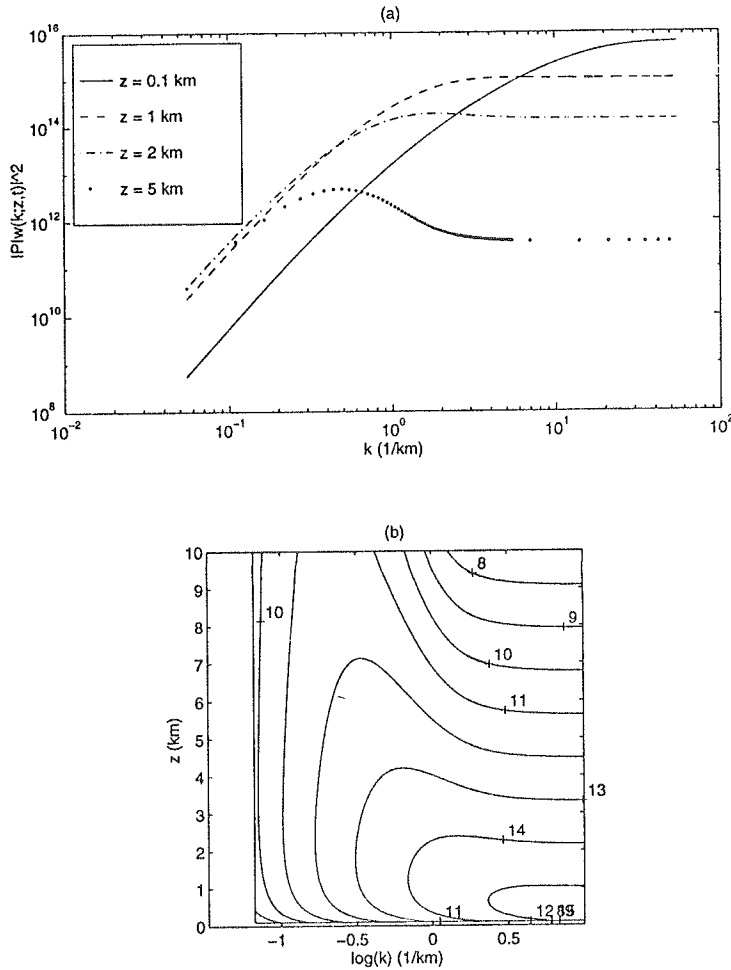


Fig. 4. (a) $|\Pi_w(k; z, t)|^2$ from (24); (b) $z - k$ distribution of $\lg(|\Pi_w|^2)$, for $N = 0$ and $u_0 = 0$ at $t = 12:00$ noon.

the surface. The stable atmosphere substantially reduces the intensity and the vertical range of the flow.

We conclude that stability N and synoptic wind u_0 inhibit the development of the flow caused by the thermal inhomogeneity of the landscape. The flow will not develop to a significant level unless in the synoptic environment with neutral stratification and weak synoptic wind.

4.2. PROPERTIES OF THE LENGTH SCALE

The length scale of the thermally induced atmospheric flow will be studied by analyzing the frequency response function (a term borrowed from *system analysis*) Π_η in equations (33). For the case of zero synoptic wind Π_w depends only on the radius wave number k , implying an isotropic field of vertical velocity w . Hence the scale analysis of the flow is simpler when studying the properties of Π_w with $u_0 = 0$.

A common feature of Π_w for all altitudes is that it goes to zero as wave number k goes to zero. This means the thermal heterogeneity with sufficiently large scale L ($= 2\pi/k$) is

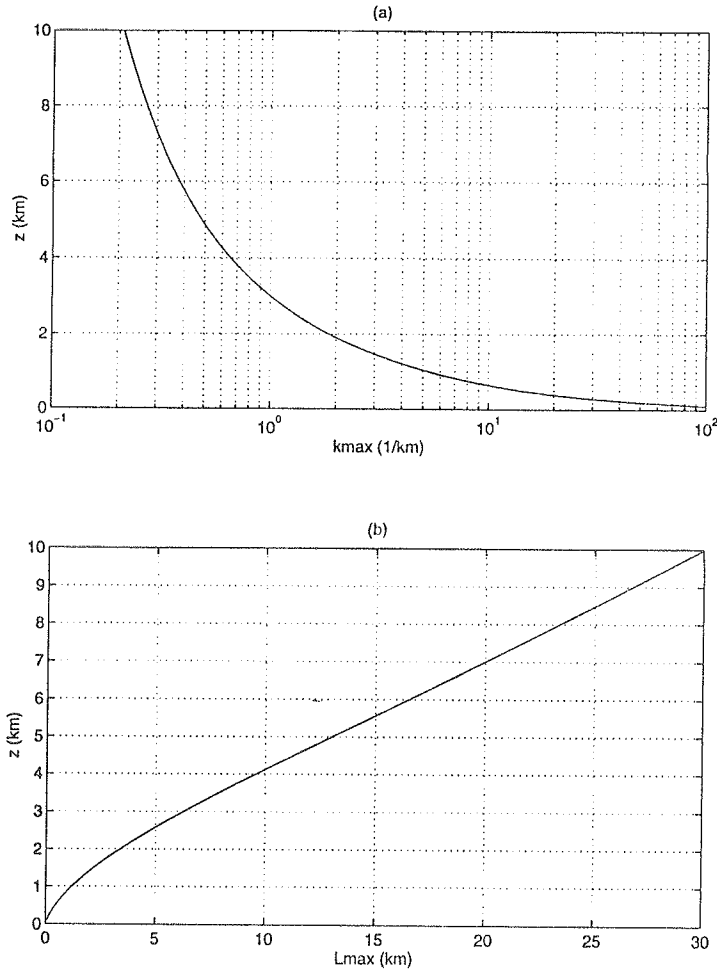


Fig. 5. (a) k_{\max} is the wave number at which $|\Pi_w|^2$ in Figure 4 reaches the maximum; (b) $L_{\max} = 2\pi/k_{\max}$ for $N = 0$ and $u_0 = 0$.

inefficient in driving the land-breeze type flow in the three-dimensional domain. On the other hand, Π_w behaves differently as a function of wave number k at different altitudes. At low altitude, for example $z = 0.1$ km, Π_w (Figure 4) initially increases with the wave number k , then saturates at $k \simeq 50$, corresponding to a length scale $L \simeq 0.1$ km. At $z = 1$ km, Π_w behaves similarly but the saturation point moves from $k \simeq 50$ down to $k \simeq 5$. Consequently the corresponding length scale L increases from 0.1 km to about 1 km. Hence at relatively low altitude Π_w does not prefer any particular wave number or length scale of the thermal forcing. However at higher altitudes, Π_w reaches a maximum at a certain finite wave number, k_{\max} . With increasing altitude, the peak of Π_w becomes sharper, and k_{\max} becomes smaller. This property of Π_w implies that the atmosphere is most sensitive to the thermal forcing at this particular wave number k_{\max} . The graph in Figure 5(a) shows the dependence of k_{\max} on altitude z . The $L_{\max} - z$ relation is plotted in Figure 5(b).

5. Conclusions

In this paper a 3D stochastic linear theory of mesoscale circulation induced by thermal heterogeneity of the land surface is developed. The governing equations of atmospheric flow are formulated as a set of linear stochastic partial differential equations (SPDE's) driven by the randomly variable diabatic heating due to the turbulent sensible heat flux. The SPDE's have been solved analytically.

The intensity of the thermally-induced flow at mesoscale is shown to be proportional to the standard deviation of the turbulent sensible heat flux at the surface. In the lower atmosphere the thermal variability of the landscape at smaller length scales is more efficient in triggering convection, while at higher altitudes the atmospheric dynamics are more sensitive to the thermal forcing at specific length scales. The atmosphere from bottom to top behaves as a low pass filter to the thermal variability of the landscape. Stable stratification and synoptic wind strongly inhibit the development of the thermally-induced mesoscale circulation.

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